

An Isoperimetric Inequality for Eigenvalues of the Bi-Harmonic Operator

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Abstract

In this article, we put forward a Neumann eigenvalue problem for the bi-harmonic operator Δ^2 on a bounded smooth domain Ω in the Euclidean n -space \mathbf{R}^n ($n \geq 2$) and then prove that the corresponding first non-zero eigenvalue $\Upsilon_1(\Omega)$ admits the isoperimetric inequality of Szegő-Weinberger type: $\Upsilon_1(\Omega) \leq \Upsilon_1(B_\Omega)$, where B_Ω is a ball in \mathbf{R}^n with the same volume of Ω . The isoperimetric inequality of Szegő-Weinberger type for the first nonzero Neumann eigenvalue of the even-multi-Laplacian operators Δ^{2m} ($m \geq 1$) on Ω is also exploited.

Keywords: Neumann problem; first eigenvalue; Min-Max principle; trial function

Mathematics Subject Classification 2000: 58J50, 35P15, 74K20, 35J40, 35R01

1 Introduction

The study of eigenvalue problems of the Laplacian operator Δ and related operators on a bounded smooth domain Ω in the Euclidean space \mathbf{R}^n or in an n -dimensional Riemannian manifold M is a topic that always attracts much attentions. The motivation comes directly from physical problems such as the membrane problem, the vibration of a clamped plate, the buckling problem and so on (for example, see [8]). The typical eigenvalue problems for the Laplacian operator Δ are

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (\text{Dirichlet Problem}) \quad (1)$$

and

$$\begin{cases} \Delta u = -\mu u, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \quad (\text{Neumann Problem}) \quad (2)$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on the boundary $\partial\Omega$ of Ω . We always assume $\partial\Omega$ is sufficiently smooth throughout the article, unless otherwise specified. One of the most important and interesting subjects in the study of eigenvalue problems is to understand the isoperimetric property of eigenvalues (see [1, 5, 6], for example). The earliest isoperimetric

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inequality for eigenvalues is the first Dirichlet eigenvalue of the Laplacian Δ (i.e. the fixed membrane) conjectured by Rayleigh in 1877 (see [15]):

$$\lambda_1(\Omega) \geq \lambda_1(B_\Omega), \quad (3)$$

for a bounded domain $\Omega \subset \mathbf{R}^n$, where B_Ω is a ball in \mathbf{R}^n with the same volume of Ω , and the equality holds if and only if Ω is the ball B_Ω . This conjecture was proved independently by Faber [9] and Krahn [10] by using the technique of symmetrization. The second isoperimetric inequality for eigenvalues is

$$\mu_1(\Omega) \leq \mu_1(B_\Omega) \quad (4)$$

for the first nonzero Neumann eigenvalue of the Laplacian Δ (i.e. the free membrane) on a bounded smooth domain Ω in \mathbf{R}^n and the equality holds if and only if Ω is the ball B_Ω . This isoperimetric inequality was proved in the case of $n = 2$ and conjectured for any $n \geq 3$ by Szegő in [17] in 1954. Two years later, Weinberger showed that the Szegő's conjecture is true in [20]. A beautiful isoperimetric inequality for eigenvalues was conjectured by Payne, Polya and Weiberger in 1956 in [13], which says

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B_\Omega)}{\lambda_1(B_\Omega)} \quad (5)$$

for the first two eigenvalues of the Dirichlet Laplacian of the Laplacian Δ on a bounded smooth domain Ω in \mathbf{R}^n and the equality holds if and only if Ω is the ball B_Ω . This conjecture was proved by Ashbaugh and Benguria in [2, 3] in 1991. One may also refer to [1] and [5] Chapter IV for the discussion of the isoperimetric property of the first (Dirichlet) eigenvalue of the Laplacian operator Δ on a normal domain in a general Riemannian manifold M , especially in a space form \mathbb{M}_κ of constant curvature κ .

On the other hand, the following Dirichlet eigenvalue problem of the bi-harmonic operator Δ^2 on a bounded domain Ω in \mathbf{R}^n is well-known:

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \quad (6)$$

When $n = 2$, Problem (6) relates to the vibration of a clamped plate in physics (see [8]). It has discrete eigenvalues

$$0 < \Gamma_1 \leq \Gamma_2 \leq \dots \leq \Gamma_k \leq \dots \nearrow \infty.$$

In 1877, Rayleigh also made a conjecture that, for the first eigenvalue $\Gamma_1(\Omega)$ of (6) on Ω in the plane, one has the following isoperimetric inequality

$$\Gamma_1(\Omega) \geq \Gamma_1(B_\Omega), \quad (7)$$

and the equality holds if and only if Ω is a disk B_Ω . This isoperimetric inequality might apply equally well on a bounded smooth domain Ω in the Euclidean n -space \mathbf{R}^n ($n \geq 3$), which is regarded as the inequality of Rayleigh-Faber-Krahn type for the first Dirichlet eigenvalue of the bi-harmonic operator Δ^2 . Conjecture (7) was proved by Nadirashvili in the case of $n = 2$ in [12] and later by Ashbaugh and Benguria in dimension $n = 3$ in [4]. However, this conjecture is still open for $n \geq 4$.

It is very natural to conjecture that the isoperimetric inequality of Szegő-Weinberger type might hold equally well for the bi-harmonic operator Δ^2 on a bounded smooth domain Ω in

the Euclidean n -space \mathbf{R}^n . In this article, we shall focus our attention on this problem. The understanding of this problem is in fact divided into two parts. One is to propose a suitable eigenvalue problem for the bi-harmonic operator Δ^2 on a bounded smooth domain Ω in \mathbf{R}^n , which is actually a generalization the Neumann eigenvalue problem (2) for the bi-harmonic Δ^2 . Another one is to prove that the isoperimetric inequality of Szegő-Weinberger type is true for the corresponding first nonzero eigenvalue.

The first problem is not difficult to answer. In fact, by using the Green's formula on a bounded smooth domain Ω in \mathbf{R}^n :

$$\int_{\Omega} (u\Delta v - \Delta u v) = \oint_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right),$$

where, as mentioned, $\frac{\partial}{\partial n}$ denotes the outward normal derivative on the boundary $\partial\Omega$ of Ω , we have

$$\int_{\Omega} \Delta u \cdot \Delta v = \int_{\Omega} u \cdot \Delta^2 v - \oint_{\partial\Omega} \left(u \frac{\partial \Delta v}{\partial n} - \frac{\partial u}{\partial n} \Delta v \right) \quad (8)$$

and

$$\int_{\Omega} \Delta u \cdot \Delta v = \int_{\Omega} \Delta^2 u \cdot v - \oint_{\partial\Omega} \left(v \frac{\partial \Delta u}{\partial n} - \frac{\partial v}{\partial n} \Delta u \right). \quad (9)$$

Therefore, besides the Dirichlet eigenvalue problem (6), the formulas (8) and (9) lead us to propose the following three possible eigenvalue problems for the bi-harmonic operator Δ^2 such that, under the boundary conditions, Δ^2 is a self-adjoint operator from $L^2(\Omega)$ to $L^2(\Omega)$:

$$\begin{cases} \Delta^2 u = \Upsilon u, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \quad (10)$$

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \quad (11)$$

and

$$\begin{cases} \Delta^2 u = \Upsilon u, & \text{in } \Omega, \\ \Delta u = \frac{\partial \Delta u}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \quad (12)$$

Among the candidates problems (10-12), only Problem (10) involves the usual Neumann condition: $\frac{\partial u}{\partial n}|_{\partial\Omega} = 0$. Furthermore, with the boundary conditions proposed by (10), it is easy to verify that the bi-harmonic operator Δ^2 is a self-adjoint elliptic differential operator from $L^2(\Omega)$ to $L^2(\Omega)$ with densely defined domain

$$\mathcal{D}(\Delta^2) = \left\{ u \in H^4(\Omega) : \frac{\partial u}{\partial n} \Big|_{\partial\Omega} = \frac{\partial \Delta u}{\partial n} \Big|_{\partial\Omega} = 0 \right\}.$$

From the standard spectral theory in functional analysis, the eigenvalue problems (10) has discrete eigenvalues

$$0 = \Upsilon_0 < \Upsilon_1 \leq \Upsilon_2 \leq \cdots \leq \Upsilon_k \leq \cdots \nearrow \infty.$$

Therefore, we settle naturally Problem (10) as the generalization the Neumann eigenvalue problem (2) for the bi-harmonic Δ^2 . What remains for us to do is now to show that the first non-zero eigenvalue $\Upsilon_1(\Omega)$ of the bi-harmonic operator Δ^2 given by the problem (10) admits the isoperimetric property:

$$\Upsilon_1(\Omega) \leq \Upsilon_1(B_\Omega), \quad (13)$$

where B_Ω is a ball in \mathbf{R}^n with the same volume of Ω , and the equality holds if and only if Ω itself is a ball B_Ω .

The aim of this paper is to prove that the isoperimetric inequality (13) does hold for any bounded smooth domain Ω in the Euclidean space \mathbf{R}^n ($n \geq 2$) and then to give an affirmative answer to our problem mentioned above. The method applied here is mainly the Min-Max principle for eigenvalues and a suitably choice of trial functions. We also display the isoperimetric inequality of Szegő-Weinberger type for the first nonzero Neumann eigenvalue of the even-multi-Laplacian operators Δ^{2m} ($m \geq 2$) on Ω in \mathbf{R}^n ($n \geq 2$). Up to the authors' knowledge, there are no other results in literature on eigenvalues of the multi-Laplacian operators than universal inequalities (see [7, 18] and the references there in). Though we believe that the similar isoperimetric inequality (13) should hold for the first nonzero Neumann eigenvalue of the odd-multi-Laplacian Δ^{2m+1} ($m \geq 1$) on a bounded domain Ω in \mathbf{R}^n , our method applied in this article does not work directly in this case.

The paper is organized as follows. In section 2, we show that the first nonzero Neumann eigenvalue of the bi-harmonic operator Δ^2 on a ball $B_R(0)$ with radius $R > 0$ in \mathbf{R}^n ($n \geq 2$) is exactly the square of the first Neumann nonzero eigenvalue of the Laplacian Δ on the ball. The n first Neumann eigenfunctions of Δ on $B_R(0)$ in \mathbf{R}^n are shown to be the first Neumann eigenfunctions of Δ^2 on the ball. The same conclusion are proved to be hold for the even-multi-Laplacian Δ^{2m} with $m \geq 1$. In section 3, we prove that the isoperimetric inequality of Szegő-Weinberger type for the first nonzero Neumann eigenvalue $\Upsilon_1(\Omega)$ of Δ^2 is true on any bounded smooth domain Ω in \mathbf{R}^n ($n \geq 2$). The similar isoperimetric inequality for the first nonzero Neumann eigenvalue of Δ^{2m} is also shown to be true. Some remarks and questions are given in this section.

2 Basic lemmas

A key step in proving the isoperimetric inequality (13) is to well understand the first nonzero Neumann eigenvalue of the bi-harmonic operator Δ^2 and its eigenfunctions on a ball in the Euclidean n -space \mathbf{R}^n . One may refer to [8] for the characterization of the Dirichlet eigenvalues of the bi-harmonic operator Δ^2 and their eigenfunctions on a disk in the plane. In this section, we shall show that the first nonzero Neumann eigenvalue $\Upsilon_1(B_R(0))$ of the bi-harmonic operator Δ^2 on the ball $B_R(0)$ with the center at the origin and radius R in \mathbf{R}^n is $\mu_1^2(R)$, where $\mu_1(R)$ is the first nonzero Neumann eigenvalue of the Laplacian Δ on $B_R(0)$.

Let $\{g_k(\omega), k \geq 0\}$ be the set of all the eigenfunctions of $\Delta_{S^{n-1}}$ on the $(n-1)$ -sphere S^{n-1} , where ω denotes the spherical coordinates in S^{n-1} . We known from [5, 8] that

$$g_0(\omega) = \text{constant}, \quad g_k(\omega) = x_k \Big|_{S^{n-1}}, \quad (k = 1, \dots, n), \dots$$

where (x_1, \dots, x_n) is the standard Euclidean coordinates on \mathbf{R}^n . Furthermore $\{g_k(\omega), k \geq 0\}$ forms an orthogonal basis in $L^2(S^{n-1})$ up to a renormalization, that is,

$$\Delta_{S^{n-1}} g_k = \lambda_k g_k, \quad k \geq 0$$

with

$$\int_{S^{n-1}} |g_k(\omega)|^2 d\omega = 1,$$

where $\{\lambda_k\}_{k=0}^\infty$ are eigenvalues of the $(n-1)$ -sphere S^{n-1} counted without multiplicity. From the standard spectral theory of spheres (see [5, 8] also), we know that $\lambda_k = -j(j+n-2)$ for some $j \geq 0$. The first few of them are

$$\lambda_0 = 0, \quad \lambda_1 = \dots = \lambda_n = -(n-1).$$

For a bounded smooth domain Ω , since Δ^2 is a densely defined self-adjoint elliptic differential operator from $L^2(\Omega)$ to $L^2(\Omega)$, we see that the first nonzero Neumann eigenvalue (10) admits the Min-Max principle (refer to [5, 8] also):

$$\Upsilon_1(\Omega) = \inf_{u, \Delta u \in L^2(\Omega), \int_{\Omega} u dx = 0} \frac{\int_{\Omega} |\Delta u|^2 dx}{\int_{\Omega} u^2 dx}. \quad (14)$$

Especially, for $\Omega = B_R(0)$, we have

$$\Upsilon_1(B_R(0)) = \inf_{u, \Delta u \in L^2(B_R(0)), \int_{B_R(0)} u dx = 0} \frac{\int_{B_R(0)} |\Delta u|^2 dx}{\int_{B_R(0)} u^2 dx}. \quad (15)$$

For $u \in C^\infty(\overline{B_R(0)})$ with $\int_{B_R(0)} u(x) dx = 0$, we have the following expansion

$$u = \sum_{k=1}^{\infty} \psi_k(r) g_k(\omega), \quad (16)$$

where (r, ω) stands for the polar coordinates of \mathbf{R}^n and

$$\psi_k(r) = \int_{S^{n-1}} u(r, \omega) g_k(\omega) d\omega \in C^\infty(0, R], \quad k \geq 1.$$

The following is a basic lemma, which determines the first nonzero Neumann eigenvalues of the bi-harmonic operator Δ^2 on a ball in \mathbf{R}^n .

Lemma 1 *The first nonzero Neumann eigenvalue of the bi-harmonic operator Δ^2 on the ball $B_R(0)$ in \mathbf{R}^n ($n \geq 2$) is: $\Upsilon_1(B_R(0)) = \mu_1^2(R)$, where $\mu_1(R)$ is the first nonzero Neumann eigenvalue of the Laplacian Δ on $B_R(0)$.*

Proof. Let $J(r) = j_k(r)$ be the Bessel-type function solving

$$r^2 J'' + (n-1)r J' + [r^2 + \lambda_k] J = 0, \quad J'(R) = 0, \quad (17)$$

and $\{\nu_{k,l}\}_{l=1}^\infty$ with $\nu_{k,1} < \nu_{k,2} < \dots$ be the non-negative zeroes of $J'(r) = j'_k(r)$. It is well-known that the first nonzero Neumann eigenvalue (2) of the Laplacian Δ on the ball $B_R(0)$ is $\mu_1(B_R(0)) = \nu_{1,1}$ (for example, see [20, 16]). Due to [19] Chapter 6 (see [11] also), for every k we have the following Bessel-Fourier expansion,

$$\psi_k(r) = \sum_{l \geq 1} c_{k,l} j_k(\nu_{k,l} r),$$

where $c_{k,l} = \int_0^R \psi(r) j_k(\nu_{k,l} r) dr / \int_0^R j_k^2(\nu_{k,l} r) dr$. Combining this with (16), we obtain that, for $u \in C^\infty(\overline{B_R(0)})$,

$$u(x) = \sum_{k \geq 1, l \geq 1} c_{k,l} j_k(\nu_{k,l} r) g_k(\omega)$$

and hence

$$\int_{B_R(0)} |u(x)|^2 dx = \sum_{k \geq 1, l \geq 1} c_{k,l}^2 \int_0^R r^{n-1} |j_k(\nu_{k,l} r)|^2 dr.$$

Furthermore, for $u \in C^\infty(\overline{B_R(0)})$ with $\int_{B_R(0)} u(x) dx = 0$ and $\partial_r u = \partial_r \Delta u = 0$ on $r = R$, we have, for any fixed $k \geq 1$ and $l \geq 1$,

$$\begin{aligned} \int_{B_R(0)} j_k(\nu_{k,l} r) g_k(\omega) \Delta u(x) dx &= \int_{B_R(0)} \Delta \left(j_k(\nu_{k,l} r) g_k(\omega) \right) u(x) dx \\ &= \int_{B_R(0)} L_k \left(j_k(\nu_{k,l} r) \right) g_k(\omega) u(x) dx \\ &= \nu_{k,l} \int_{B_R(0)} j_k(\nu_{k,l} r) g_k(\omega) u(x) dx, \end{aligned} \quad (18)$$

where

$$L_k = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{\lambda_k}{r^2}.$$

Here in the last equality of (18) we have used the following identity

$$\begin{aligned} \Delta \left(j_k(\nu_{k,l} r) g_k(\omega) \right) &= \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}} \right) \left(j_k(\nu_{k,l} r) g_k(\omega) \right) \\ &= g_k(\omega) L_k \left(j_k(\nu_{k,l} r) \right) \\ &= \nu_{k,l} j_k(\nu_{k,l} r) g_k(\omega). \end{aligned}$$

Hence (18) leads to, for any $u \in C^\infty(\overline{B_R(0)})$ with $\int_{B_1(0)} u(x) dx = 0$,

$$\begin{aligned} \int_{B_R(0)} |\Delta u|^2 dx &= \sum_{k \geq 1, l \geq 1} c_{k,l}^2 (\nu_{k,l})^2 \int_0^R r^{n-1} |j_k(\nu_{k,l} r)|^2 dr \\ &\geq \nu_{1,1}^2 \int_{B_R(0)} |u|^2 dx \end{aligned} \quad (19)$$

and the equality holds if and only if $u(x) = j_1(\nu_{1,1} r) \frac{x_i}{r}, i = 1, \dots, n$. By the Min-Max principle (14), we see that

$$\Upsilon_1(B_R(0)) = \nu_{1,1}^2 = \mu_1^2(R).$$

This proves Lemma 1. \square

By the way, from the proof of Lemma 1 we also have

Corollary 1 *The first (nonzero) Neumann eigenfunctions of the Laplacian Δ on the ball $B_R(0)$:*

$$j_1(\nu_{1,1} r) \frac{x_i}{r}, \quad i = 1, \dots, n,$$

where $j_1(r)$ solves (17) with $\lambda_k = \lambda_1 = -(n-1)$, are first (nonzero) Neumann eigenfunctions of the bi-harmonic operator Δ^2 on $B_R(0)$.

Up to the author's knowledge, besides universal inequalities on eigenvalues of the multi-Laplacian operators (see [7, 18] and the references there in), there are no other results in literature. In order to understand the isoperimetric property of eigenvalues of the multi-Laplacian operators, we may consider the following Neumann eigenvalue problem of the even-multi-Laplacian operator Δ^{2m} ($m \geq 1$) on a bounded domain Ω in \mathbf{R}^n :

$$\begin{cases} \Delta^{2m} u = \hat{\Upsilon} u, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = \dots = \frac{\partial \Delta^{2m-1} u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \quad (20)$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on the boundary $\partial\Omega$ of Ω . One notes that, when $m = 1$, Problem (20) reduces to Problem (10). It is obvious that Problem (20) has discrete eigenvalues

$$0 = \hat{\Upsilon}_0 < \hat{\Upsilon}_1 \leq \hat{\Upsilon}_2 \leq \dots \leq \hat{\Upsilon}_k \leq \dots \nearrow \infty.$$

Now we similarly establish the following lemma.

Lemma 2 *The first nonzero Neumann eigenvalue of the even-multi-Laplacian operator Δ^{2m} on the ball $B_R(0)$ in \mathbf{R}^n ($n \geq 2$) is: $\hat{\Upsilon}_1(B_R(0)) = \mu_1^{2m}(R)$, where $\mu_1(R)$ is the first nonzero Neumann eigenvalue of the Laplacian Δ on $B_R(0)$.*

Proof. We go along the line of the proof of Lemma 1. Notice that, for $u \in C^\infty(\overline{B_R(0)})$ with $\int_{B_R(0)} u(x) dx = 0$ and $\partial_r u = \partial \Delta u = \dots = \partial_r \Delta^{m-1} u = 0$ on $r = R$, we have the expansion

$$u(x) = \sum_{k \geq 1, l \geq 1} c_{k,l} j_k(\nu_{k,l} r) g_k(\omega)$$

and, for any fixed $k \geq 1$ and $l \geq 1$,

$$\int_{B_R(0)} j_k(\nu_{k,l} r) g_k(\omega) \Delta^m u(x) dx = (\nu_{k,l})^m \int_{B_R(0)} j_k(\nu_{k,l} r) g_k(\omega) u(x) dx.$$

This leads to

$$\begin{aligned} \int_{B_R(0)} |\Delta^m u|^2 dx &= \sum_{k \geq 1, l \geq 1} c_{k,l}^2 (\nu_{k,l})^{2m} \int_0^R r^{n-1} |j_k(\nu_{k,l} r)|^2 dr \\ &\geq (\nu_{1,1})^{2m} \int_{B_R(0)} |u|^2 dx, \end{aligned} \quad (21)$$

and the equality holds if and only if $u(x) = j_1(\nu_{1,1} r) \frac{x_i}{r}$, $i = 1, \dots, n$. Now by the Min-Max principle for the first eigenvalue $\hat{\Upsilon}_1(B_R(0))$ of Problem (20):

$$\hat{\Upsilon}_1(B_R(0)) = \inf_{u, \Delta^m u \in L^2(B_R(0)), \int_{B_R(0)} u dx = 0} \frac{\int_{B_R(0)} |\Delta u|^{2m} dx}{\int_{B_R(0)} u^2 dx},$$

we have

$$\hat{\Upsilon}_1(B_R(0)) = \nu_{1,1}^{2m} = \mu_1^{2m}(R).$$

This proves Lemma 2. \square

Corollary 2 *The first (nonzero) Neumann eigenfunctions of the Laplacian Δ on the ball $B_R(0)$:*

$$j_1(\nu_{1,1}r)\frac{x_i}{r}, \quad i = 1, \dots, n,$$

where $j_1(r)$ solves (17) with $\lambda_k = \lambda_1 = -(n-1)$, are first (nonzero) Neumann eigenfunctions of the bi-harmonic operator Δ^{2m} on $B_R(0)$.

Based on the basic lemma 1 or lemma 2, the proof of the isoperimetric inequality (13) of Szegő-Weinberger type for the first nonzero Neumann eigenvalue of Δ^2 or Δ^{2m} becomes now clear and natural. This will be done in the next section.

3 Isoperimetric inequality for the Neumann eigenvalue of the bi-harmonic operator

In this section, we shall first prove that the isoperimetric inequality (13) of Szegő-Weinberger type holds still for the first nonzero Neumann eigenvalue $\Upsilon_1(\Omega)$ of the bi-harmonic operator Δ^2 . Then we generalize it to the even-multi-Laplacian operator Δ^{2m} on a bounded Ω in \mathbf{R}^n .

Theorem 1 *Let Ω be a connected bounded smooth domain in \mathbf{R}^n ($n \geq 2$), then the first nonzero Neumann eigenvalue $\Upsilon_1(\Omega)$ of the bi-harmonic operator Δ^2 satisfies the following inequality of Szegő-Weinberger type:*

$$\Upsilon_1(\Omega) \leq \Upsilon_1(B_\Omega), \quad (22)$$

where B_Ω is a ball in \mathbf{R}^n with the same volume of Ω . Moreover, the equality holds if and only if Ω is B_Ω .

Proof. By the Min-Max principle, one has

$$\Upsilon_1(\Omega) = \inf_{u, \Delta u \in L^2(\Omega), \int_\Omega u = 0} \frac{\int_\Omega |\Delta u|^2 dx}{\int_\Omega |u|^2 dx}.$$

We try to construct some suitable trial functions u with $\int_\Omega u = 0$ to give an optimal upper bound for $\Upsilon_1(\Omega)$. From the basic lemma 1 in the previous section, we know that

$$g(r)\frac{x_i}{r}, \quad i = 1, \dots, n$$

are eigenfunctions corresponding to $\Upsilon_1(B_\Omega)$ of the bi-harmonic operator Δ^2 on the ball B_Ω , where $g(r)$ satisfies

$$\begin{cases} \frac{d^2 g}{dr^2} + \frac{n-1}{r} \frac{dg}{dr} + \left(\mu_1 - \frac{n-1}{r^2}\right) g = 0 \\ \frac{dg}{dr}(R) = 0, \end{cases} \quad (23)$$

where μ_1 is the first nonzero Neumann eigenvalue of Δ on B_Ω and R is the radius of B_Ω , i.e. $B_\Omega = B_R(0)$. If we let $\phi(r) = r^{\frac{n-2}{2}} g(r)$, then we transform Eq.(23) into

$$\phi'' + \frac{1}{r} \phi' + \left(\mu_1 - \frac{\left(\frac{n}{2}\right)^2}{r^2}\right) \phi = 0,$$

which is a standard Bessel equation:

$$\varphi'' + \frac{1}{r}\varphi' + \left(1 - \frac{\left(\frac{n}{2}\right)^2}{r^2}\right)\varphi = 0,$$

by a rescaling $r \rightarrow \sqrt{\mu_1} r$. This implies that $g(r)$ can be explicitly given by

$$g(r) = r^{-\frac{n-2}{2}} J_{\frac{n}{2}}(\sqrt{\mu_1} r),$$

where $J_{\frac{n}{2}}(r)$ is the $\frac{n}{2}$ -Bessel function of first kind and $\sqrt{\mu_1}$ is the first positive zero of $g(r)$. Now we define an auxiliary function

$$G(r) = g(r), \quad r \geq 0.$$

For any $x_0 \in$ the convex hull of Ω , we define a vector by

$$V(x_0) = \sum_{i=1}^n \left(\int_{\Omega} \frac{(x - x_0)G(r(x, x_0))}{r(x, x_0)} \right) \frac{\partial}{\partial x_i},$$

which can be regarded as a continuous vector field on the convex hull of Ω . Furthermore, it is easy to see that $V(x_0)$ points inward at the boundary of the convex hull of Ω . Therefore, it follows from the Hopf theorem that there exists an x^0 in the convex hull of Ω such that $V(x^0) = 0$ (the detailed arguments are referred to [20]). By a translation in \mathbf{R}^n , we may assume that $x^0 = 0$ without the loss of generality and thus set $u_i = G(r)\frac{x_i}{r}, i = 1, \dots, n$. Then we have

$$\int_{\Omega} u_i = 0, \quad i = 1, \dots, n.$$

We will see below that $u_i, \Delta u_i \in L^2(\Omega)$ ($i = 1, \dots, n$) by our above construction of $G(r)$. Therefore, for the bounded smooth domain Ω in \mathbf{R}^n , by the Min-Max principle we have

$$\Upsilon_1(\Omega) \int_{\Omega} \sum_{i=1}^n u_i^2 \leq \int_{\Omega} \sum_{i=1}^n |\Delta u_i|^2. \quad (24)$$

One also notes that

$$\begin{aligned} \sum_i |\Delta u_i|^2 &= \sum_{i=1}^n \left| \left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} + \frac{\Delta_{\mathbf{S}^{n-1}(1)}}{r^2} \right) u_i \right|^2 \\ &= \sum_{i=1}^n \left| \left[\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{n-1}{r^2} \right) G(r) \right] \frac{x_i}{r} \right|^2 \\ &= \left[\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{n-1}{r^2} \right) G(r) \right]^2, \end{aligned}$$

and $\sum_i u_i^2 = G^2$. Substituting them into (24) we have

$$\Upsilon_1(\Omega) \leq \frac{\int_{\Omega} \left[\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{n-1}{r^2} \right) G(r) \right]^2}{\int_{\Omega} G^2}. \quad (25)$$

Since $G(r) = g(r)$ and $g(r)$ solves (23), we see that

$$\left[\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{n-1}{r^2} \right) G(r) \right]^2 = \mu_1^2 G^2(r).$$

Substituting this identity into (25) we finally have

$$\Upsilon_1(\Omega) \leq \frac{\int_{\Omega} \left[\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{n-1}{r^2} \right) G(r) \right]^2}{\int_{\Omega} G^2} = \frac{\mu_1^2 \int_{\Omega} G^2}{\int_{\Omega} G^2} = \mu_1^2 = \Upsilon_1(B_{\Omega}).$$

When the equality holds, all the inequalities in the above steps become equalities, which implies that the n trial functions $u_i(x)$ ($1 \leq i \leq n$) should be eigenfunctions corresponding to $\Upsilon_1(\Omega)$. This leads to that Ω must be the ball B_{Ω} . The proof of Theorem 1 is completed. \square

We remark that our method used in the proof of Theorem 1 is applicable to the even-multi-Laplacian operator Δ^{2m} . That says, we similarly have

Theorem 2 *Let Ω be a connected bounded smooth domain in \mathbf{R}^n ($n \geq 2$), then the first nonzero Neumann eigenvalue $\hat{\Upsilon}_1(\Omega)$ of the multi-Laplacian operator Δ^{2m} satisfies the following isoperimetric inequality of Szegő-Weinberger type:*

$$\hat{\Upsilon}_1(\Omega) \leq \hat{\Upsilon}_1(B_{\Omega}). \quad (26)$$

Moreover, the equality holds if and only if Ω is B_{Ω} .

Proof. For the bounded smooth domain Ω in \mathbf{R}^n , by the Min-Max principle, one has

$$\hat{\Upsilon}_1(\Omega) = \inf_{f, \Delta^m f \in L^2(\Omega), \int_{\Omega} f = 0} \frac{\int_{\Omega} |\Delta^m f|^2}{\int_{\Omega} |f|^2}. \quad (27)$$

Now we try to construct some suitable trial functions f to give an optimal upper bound for $\hat{\Upsilon}_1(\Omega)$. From the basic lemma 2 in the previous section, we know that

$$g(r) \frac{x_i}{r}, \quad i = 1, \dots, n$$

are first eigenfunctions of Δ^{2m} on B_{Ω} , where $g(r) = r^{-\frac{n-2}{2}} J_{\frac{n}{2}}(\sqrt{\mu_1} r)$ satisfies (23), in which $J_{\frac{n}{2}}(r)$ is the $\frac{n}{2}$ -Bessel function of first kind. With the same argument done in the proof of Theorem 1, up to a translation in \mathbf{R}^n , we may have

$$\int_{\Omega} f_i = 0, \quad i = 1, \dots, n,$$

where $f_i = g(r) \frac{x_i}{r}$, $i = 1, \dots, n$. It is also obvious that $f_i, \Delta^m f_i \in L^2(\Omega)$ ($i = 1, \dots, n$). By the Min-Max principle (27) we see that

$$\mu_1(\Omega) \int_{\Omega} \sum_{i=1}^n f_i^2 \leq \int_{\Omega} \sum_{i=1}^n |\Delta^m f_i|^2. \quad (28)$$

Now

$$\sum_i |\Delta^m f_i|^2 = \sum_{i=1}^n \left| \left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} + \frac{\Delta_{\mathbf{S}^{n-1}(1)}}{r^2} \right)^m f_i \right|^2$$

$$\begin{aligned}
&= \sum_{i=1}^n \left| \left[\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{n-1}{r^2} \right)^m g(r) \right] \frac{x_i}{r} \right|^2 \\
&= \left[\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{n-1}{r^2} \right)^m g(r) \right]^2,
\end{aligned}$$

and $\sum_i f_i^2 = g^2$. Substituting these identities into (28) we have

$$\hat{\Upsilon}_1(\Omega) \leq \frac{\int_{\Omega} \left[\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{n-1}{r^2} \right)^m g(r) \right]^2}{\int_{\Omega} g^2}. \quad (29)$$

Since $g(r)$ satisfies (23), we have that the integrand

$$\begin{aligned}
Q(r) &= \left[\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{n-1}{r^2} \right)^m g(r) \right]^2 \\
&= (-\mu_1)^{2m} g^2(r) = \mu_1^{2m} g^2(r).
\end{aligned} \quad (30)$$

Substituting (30) into (29) we finally have

$$\hat{\Upsilon}_1(\Omega) \leq \frac{\int_{\Omega} \left[\left(\frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{n-1}{r^2} \right) g(r) \right]^2}{\int_{\Omega} g^2} = \frac{\mu_1^{2m} \int_{\Omega} g^2}{\int_{\Omega} g^2} = \mu_1^{2m} = \hat{\Upsilon}_1(B_{\Omega}).$$

With the same argument in the proof of Theorem 1, we also obtain $\Omega = B_{\Omega}$ when the equality holds. This completes the proof of Theorem 2. \square

Theorem 1 indicates that not only the eigenvalue problem (10) is actually the Neumann eigenvalue problem for the bi-harmonic operator Δ^2 on a bounded smooth domain Ω in \mathbf{R}^n , but also the isoperimetric inequality of Szegő-Weinberger type for the first nonzero eigenvalue $\Upsilon_1(\Omega)$ holds still for the bi-harmonic operator. Theorem 2 shows that the similar result is true for the even-multi-Laplacian Δ^{2m} ($m \geq 1$). This might reveal a universal isoperimetric property of the first nonzero Neumann eigenvalue of the multi-Laplacian Δ^m ($m \geq 2$).

Finally we point out that whether the isoperimetric inequality (13) holds for the first nonzero Neumann eigenvalue of the odd-multi-Laplacian Δ^{2m+1} is unknown at present time, since our method applied here does not work directly in this case. However, we conjecture that it is true. Another interesting problem is: Whether all the first eigenfunctions of the bi-harmonic operator Δ^2 on the ball B_{Ω} are just these listed in Corollary 1 or not. These problems deserve for future study.

Acknowledgement

The authors are supported by the National Natural Science Foundation of China (Q.D. by Grant Nos.10971030, F.G. by J0730103 and Y.Z. by Grant No. 11031001).

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